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USE OF FUZZY NUMBERS FOR ASSESSING THE ACQUISITION OF THE VAN-HIELE LEVELS IN GEOMETRY

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Problem formulation. The pedagogical value of the Euclidian geometry is great, mainly because it cultivates the student cognitive skills and connects directly mathematics to the real world. However, students face many difficulties for the learning of the Euclidian geometry, which fluctuate from the understanding of the space to the development of geometric reasoning and the ability of constructing the proofs and solutions of several geometric propositions and problems.

The van Hiele (vH) theory of geometric reasoning [12, 13] suggests that students can progress through five levels of increasing structural complexity. A higher level contains all knowledge of any lower level and some additional knowledge which is not explicit at the lower levels, Therefore, each level appears as a meta-theory of the previous one [2]. The five vH levels include:

- L_1 ***(Visualization):*** Students perceive the geometric figures as entities according to their appearance, without explicit regard to their properties.
- L_2 ***(Analysis):*** Students establish the properties of geometric figures be means of an informal analysis of their component parts and begin to recognize them by their properties.
- L_3 ***(Abstraction):*** Students become able to relate the properties of figures, to distinguish between the necessity and sufficiency of a set of properties in determining a concept and to form abstract definitions.
- L_4 ***(Deduction):*** Students reason formally within the context of a geometric system and they gasp the significance of deduction as means of developing geometric theory.
- L_5 ***(Rigor):*** Students understand the foundations of geometry and can compare geometric systems based on different axioms.

Obviously the level L_5 is very difficult, if not impossible, to appear in secondary classrooms, while level L_4 also appears very rarely.

Although van Hiele [13] claimed that the above levels are discrete – which means that the transition from a level to the next one does not happen gradually but all at once – alternative researches by Burnes & Shaughnessy [1], Fuys et al. [3], Wilson [17], Guttierrez et al. [4] and by Perdikaris [7] suggest that the vH levels are ***continuous*** characterized by transitions between the adjacent levels. This means that from the teacher's point of view there exists fuzziness about the degree of student acquisition of each vH level. Therefore, principles of ***Fuzzy Logic (FL)*** could be used for the assessment of student geometric reasoning skills.

Perdikaris [6] presented a fuzzy framework for the vH theory by introducing a finite absorbing Markov chain [5] on the fuzzy linguistic labels (characterizations) of no, low, intermediate, high and complete acquisition respectively of each vH level. He also used [8] the fuzzy possibilities of student profiles and the generalization in possibility theory of the classical Shannon's entropy for measuring a system's uncertainty to compare the intelligence of student groups in the vH level theory. However, this method needs laborious calculations in its final application.

In this paper an alternative method of FL is used for the assessment of student acquisition of the vH levels, by using the ***Triangular Fuzzy Numbers (TFNs)***. This method is very simple in its final application and its outcomes can be easily interpreted. The rest of the paper is formulated as follows: In the second Section the background on TFNs which is necessary for the paper's understanding is recalled. In the third Section the utilization of TFNs as an assessment tool is reviewed, while in the fourth Section an example is presented illustrating its applicability in the vH level theory. Finally, the fifth and last Section is devoted to the conclusion and to some hints for future research.

Triangular fuzzy numbers (TFNs). For general facts on ***Fuzzy Sets (FS)*** and ***Fuzzy Numbers (FNs)*** we refer to [16]. It is recalled here that a FN is a FS on the set ***R*** of the real numbers which is normal (i.e. there exists *x* in ***R*** such that the membership degree *m(x)* = 1) and convex (i.e. all it's a-cuts, with a in [0, 1], are closed real intervals), while its membership function *y = m(x)* is piecewise continuous.

It is also recalled that a TFN (*a, b, c*), with *a, b, c* real numbers such that *a<c<b*, is a FN with membership function defined by

$$y = m(x) = \begin{cases} \frac{x-a}{b-a}, & x \in [a,b] \\ \frac{c-x}{c-b}, & x \in [b,c] \\ 0, & x < a \text{ or } x > c \end{cases}$$

Let A (*a, b, c*) and B (a_1, b_1, c_1) be two TFNs and let *k* be a positive real number. Then the ***sum*** A + B=($a+a_1$, $b+b_1$, $c+c_1$) and the ***scalar product*** *kA*=(*ka, kb, kc*) ([16], paragraph 10). Further, given the TFNs A_i, i = 1, 2,…, *n* , where *n* a non negative integer, $n \geq 2$, their ***mean value*** is defined to be the TFN A= ($A_1 + A_2 + \ldots + A_n$)/n.

Assessment of a student group performance using TFNs. First**,** the individual student performance is numerically evaluated in a climax from 0 to 100. In order to characterize it qualitatively we have introduced in [16] the fuzzy linguistic labels (grades) A (85-100) = excellent, B (75-84) = very good, C (60-74) = good, D(50-59) = fair and F(0-49) = non satisfactory. We have also assigned to each of the above grades a TFN denoted by the same letter as follows: A=(85, 92.5, 100), B=(75, 79.5, 84), C=(60, 67, 74), D=50, 54.5, 59) and F=(0, 24,5, 48). The middle entry of each of those TFNs is equal to the mean value of the student scores attached to the corresponding grade. In this way a TFN can be assigned to each student assessing his/her individual performance. Therefore, it is logical to use the mean value M of all those TFNs for evaluating the student group overall performance.

In [16] we have used the ***Center of Gravity (COG)*** technique for the defuzzification of the TFNs. This technique leads to the representation of a TFN T=(*a, b, c*) by the x-coordinate, say x(T), of the COG of its graph. Since the graph is a triangle ABC with A(a, 0), B(1, b) and C(0,c) ([16], Figure 2), x(T)=(a+b+c)/3 (1) ([16], Proposition 8].

In particular, if T is one of the TFNs A, B, C, D, F then b=(a+c)/2. Therefore, equality (1) gives that

$$x(T) = \frac{a + \frac{a+c}{2} + c}{3} = \frac{3(a+c)}{6} = b.$$

But M=k_1A+k_2B+k_3C+k_4D+k_5F, with k_i non negative real numbers, *i*=1, 2, 3, 4, 5. Therefore, if M(*a, b, c*), then obviously x(M)= $k_1x(A)+k_2x(B)+k_3x(C)+k_4x(D)+k_5x(F)=b$

REMARK: An alternative way to defuzzify a TFN T=(*a, b, c*) is to use the ***Yager Index*** Y(T), introduced in [18] in terms of the a-cuts of T, a in [0, 1], in order to help the ordering of fuzzy sets. It can be shown ([9], p. 62) that Y(T)=(2b+a+c)/4.

Observe now that x(T)=Y(T) $\Leftrightarrow$ (a+b+c)/3=(2b+a+c)/4 $\Leftrightarrow$ 4(*a*+*b*+*c*)=3(2*b*+*a*+*c*) $\Leftrightarrow$ *a*+*c*=2*b*. The last equality is not true in general for *a<b<c;* e.g. take *a*=1, *b*=2.5 and *c*=3. In other words in general we have that x(T) $\neq$ Y(T). However, since the middle entries of the TFNs A, B, C, D, and F have been defined to be equal to the mean values of their other two entries, the above equality holds for those TFNs. Therefore, since the mean value M is a linear combination of the TFNs A, B, C, D, and F, it is straightforward to check that x(M)=Y(M). Thus, the above two defuzzification methods provide the same assessment outcomes when used with those TFNs..

Assessing the acquisition of the van Hiele levels. Gutierrez et al. [4] presented a paradigm for evaluating the acquisition of the vH levels in three–dimensional Geometry by three different groups, say G_1, G_2 and G_3, consisting of 20, 21 and 9 students respectively. Here, in order to illustrate the utilization of TFNs as an assessment tool in the vH level theory we shall use the data of this paradigm, which are depicted in Table 1.

Table 1

Data of the paradigm **[Degree of acquisition]**

Group	vH level	F	D	C	B	A
G_1	L_1	0	0	0	0	20
G_1	L_2	1	0	3	6	10
G_1	L_3	2	3	6	6	3
G_2	L_1	0	0	1	2	18
G_2	L_2	0	3	4	13	1
G_2	L_3	9	6	5	1	0
G_3	L_1	0	2	4	2	1
G_3	L_2	3	4	2	0	0
G_3	L_3	9	0	0	0	0

From the data of the first row of Table 1 one obtains that M=A and x(M)=92.5 for G1 in L1. Similarly, from the second row it is obtained that M= (F+3C+6B+10A)/20=(74, 81.38, 88.75) and x(M)=81.38 for G1 in L2. Therefore the first group demonstrates a very good (B) overall performance in level L2. We keep going in the same way with the remaining rows. All the assessment outcomes of the paradigm are depicted in Table 2.

Table 2

Assessment outcomes

Group	vH level	M	x(M)	Group's performance
G_1	L_1	(85, 92.5, 100)	92.5	A
G_1	L_2	(74, 81.38, 88.75)	81.38	B
G_1	L_3	(60.75, 68.45, 76.15)	68.45	C
G_2	L_1	(82.86, 90.05, 97.24)	90.05	A

Group	vH level	M	x(M)	Group's performance
G_2	L_2	(69.05, 74.17, 79.69)	74.37	C
G_2	L_3	(32.14, 45.81, 59.48)	45.81	F
G_3	L_1	(63.89, 69.83, 75.78)	69.84	C
G_3	L_2	(35.56, 47.28, 59)	47.28	F
G_3	L_3	(0, 24.5, 59)	24.5	F

Observing the values of x(M) in Table 2 it becomes evident that G_1 demonstrates the best performance in all levels, followed by G_2 and G_3. Further, from the last column of Table 2 it turns out that. G_1 demonstrates excellent performance in L_1, very good in L_2 and good in L_3, G_2 demonstrates excellent performance in L_1, good in L_2 and non satisfactory in L_3, while G_3 demonstrates good performance in L_1 and non satisfactory performance in L_2 and L_3. Finally, it is logical to evaluate the overall performance of each group in the first three vH levels by calculating the mean value of its performances in each level. Thus, G_1 demonstrated a very good ($\frac{92.5+81.38+68.45}{3} \approx 80.78$), G_2 demonstrated a good ($\frac{90.05+7437+45.81}{3} \approx 70.08$) and G_3 demonstrated a non satisfactory ($\frac{69.84+47.28+24.5}{3} \approx 47.21$) overall performance.

Conclusion and hints for future research. In this work we utilized TFNs for assessing the acquisition of the vH levels by students. The application of this method is very simple and its outcomes can be easily interpreted. Other fuzzy assessment methods have been also used in earlier works by the author, his research collaborator Prof. I. Subbotin and others, including the measurement of a system's uncertainty ([8, 14], etc.) and the application of the COG defuzzification technique and its variations ([10, 11, 15], etc.). Our plans for future research include the effort to compare the outcomes of all these methods in order to obtain useful conclusions about their advantages and disadvantages.

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Abstract. Voskoglou M.Gr. Use of Fuzzy Numbers for Assessing the Acquisition of the van-Hiele Levels in Geometry.

It is generally accepted that students face many difficulties in constructing proofs of theorems and solutions of problems of the Euclidean Geometry. The van Hiele theory of geometric reasoning suggests that students can progress through five levels of increasing structural complexity. It has been also proved by other researchers that these levels are continuous characterized by transitions between the successive levels. This means that from the teacher's point of view there exists fuzziness about the degree of student acquisition of each level, which suggests that principles of Fuzzy Logic could be used for the evaluation of student geometric reasoning skills. Here a combination of Triangular Fuzzy Numbers (TFNs) and of the Centre of Gravity (COG) deffuzzification technique is applied for evaluating the acquisition of the van Hielie levels by students. It is further shown that the use of the Yager index instead of the COG technique leads to the same assessment conclusions. Other fizzy methods applied in *earlier works are also discussed and an example on high school student acquisition of 3-dimensional geometrical concepts is presented illustrating our results.*

Key words: *van Hiele (vH) levels of geometric reasoning, fuzzy logic (FL), triangular fuzzy numbers (TFNs), centre of gravity (COG) defuzzification technique, Yager index.*

Аннотация. Воскоглой М. Гр. Использование нечетких чисел для оценки усвоения уровней ван Хиеле в геометрии*. Известно, что студенты сталкиваются со многими трудностями в построении доказательства теорем и решения задач евклидовой геометрии. Теория ван Хиеле из предполагает, что студенты проходят пять уровней возрастающей структурной сложности. Было доказано другими исследователями, что эти уровни характеризуются непрерывным переходами между последовательными уровнями. Естественно, существует некоторая нечеткость в представлении учителя о степени усвоения студентами каждого уровня, что свидетельствует о том, что принципы нечеткой логики можно было бы использовать для оценки студенческих геометрических навыков мышления. Комбинация методов треугольных нечетких чисел (TFNs) и Центра Тяжести (COG) применимы здесь для оценки. Показано, что использование индекса Яджера вместо метода COG приводит к тем же выводам. Представлены примеры иллюстрирующий наши результаты.*

Ключевые слова: *ван Хиеле уровни в геометрии, нечеткая логика, треугольные нечеткие числа, метод дефаззификации, индекс Ягер.*

Анотація. Воскоглой М. Гр. Використання нечітких чисел для оцінки засвоєння рівнів ван Хіеле в геометрії. *Відомо, що студенти стикаються з багатьма труднощами в побудові доведення теорем та розв'язання задач евклідової геометрії. Теорія ван Хіеле з передбачає, що студенти проходять п'ять рівнів зростаючої структурної складності. Було доведено іншими дослідниками, що ці рівні характеризуються безперервним переходами між послідовними рівнями. Природно, існує деяка нечіткість у поданні вчителя про ступінь засвоєння студентами кожного рівня, що свідчить про те, що принципи нечіткої логіки можна було б використовувати для оцінки студентських геометричних навичок мислення. Комбінація методів трикутних нечітких чисел (TFNs) і Центру Ваги (COG) застосовні тут для оцінки. Показано, що використання індексу Яджера замість методу COG призводить до тих самих висновків. Представлені приклади ілюструє наші результати.*

Ключові слова: *ван Хіеле рівні в геометрії, нечітка логіка, трикутні нечіткі числа, метод дефаззіфікації, індекс Ягер.*