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COMPUTER MODELS FOR POLYGONAL NUMBERS INVESTIGATION AND THEIR USE IN MATHEMATICAL EDUCATION

Introduction. Various education studies conducted recently across Europe confirm decreasing interest to mathematics and lack of students' motivation to master it as math seems to be too abstract for students, too far from real life tasks. It is also detected students' disinterest in advanced mathematics and science degrees. On the other hand, integrative tendencies between knowledge areas obviously are getting really strong. Nowadays social demand is to provide global labor market with high level specialists who are able to apply their mathematical thinking to practical problems solving. Mathematics as a science always has been universal language and powerful instrument for investigations in many areas. However, its cognitive facilities are used insufficiently to stimulate students' interest to science. Thus, we can distinguish two main contradictions: between urgent needs of society to raise level of mathematical education and falling students' motivation to learn mathematics deeply; between great cognitive power of mathematics as a science and insufficient using of this power in the educational process.

In order to resolve these contradictions it is necessary to propose relevant didactic mechanisms, to apply computer technologies and make mathematical concepts serve educational purposes such as creating students' mathematical outlook and culture, demonstrating them practical value of mathematics, and raising their interest to it. In this context the phenomenon of polygonal numbers seems to be really bright example of both cognitive resource of math and its application to didactics and real life as well.

Actually, we can see polygonal numbers in nature and daily life everywhere (for instance, honeycombs, spider's web, packed eggs, billiard-balls packed into a box, topology of network, etc. are everyday examples of polygonal numbers of different angularity). However, these numbers as a mathematical notion arose long before our era as a result of formation of the natural numbers series composed from the sums of arithmetic sequences. Ancient scientists gave elements of these series this or that geometrical interpretation. In particular, they considered them as sequences of regular polygons made of dots [5; 8]. Among ancient scientists who wrote about polygonal numbers we can name Greek mathematician Hypsicles of Alexandria (II century before AD), Diophantus of Alexandria (III century, AD). Some dependences between numbers of different angularity were obtained by Nicomachus of Gerase (I century, AD). Later relations among natural and polygonal numbers were investigated by mathematicians Pierre de Fermat (1601-1665), Leonhard Euler (1707-1783), Augustin Louis Cauchy (1789-1857).

However, the analysis of scientific and popular scientific literature testifies that a lot of interesting relations between flat polygonal numbers did not obtain detailed research, and some known relations can be generalized and proved by modern mathematical methods. On the other hand, polygonal numbers as a mathematical concept obviously have significant educational value: they allow students to realize a link between geometrical and algebraic essence of a number, to trace the continuity of geometry and algebra methods in mathematics, to see applications of an abstract notion to practical problems solving.

Thus, investigation of various properties of polygonal numbers as well as finding didactic ways to apply their pedagogical potential seems to be urgent and topical. So, the paper focuses on some main objectives: (1) to obtain and represent some relations among flat polygonal numbers based on their properties investigation; (2) to determine binary algebraic operations on the sets of polygonal numbers of various angularity and to investigate algebraic properties of the polygonal numbers sets as algebraic systems; (3) to represent author's computer models of polygonal numbers and the ways in which these and other results may be used in mathematical education.

Methods and instruments. Let us consider sequence of the regular triangles composed from dots. If we agree to consider the first left dot to be also a triangle, then in Figure 1 four members of this sequence are represented. We will put in correspondence to each triangle the number expressing the whole amount of dots in it. In such a way we obtain numerical sequence: 1,3,6,10,15... These are triangular numbers. If to designate them $\mathbf{3_n}$, where *n* is the index of a member of sequence, and digit **3** specifies that numbers are triangular, it is obvious from the figure (Fig.1) that $3_1=1$; $3_2=1+2$; $3_3=1+2+3$; $3_4=1+2+3+4$ etc. Obviously, the *n-th* triangular number is the sum of *n* first natural numbers, i.e. $3_n=1+2+3+...+n = n(n+1)/2$.

Quadrates (Fig. 2) and regular pentagons are similarly considered. Thus, it is possible to receive any polygonal numbers in such a way.

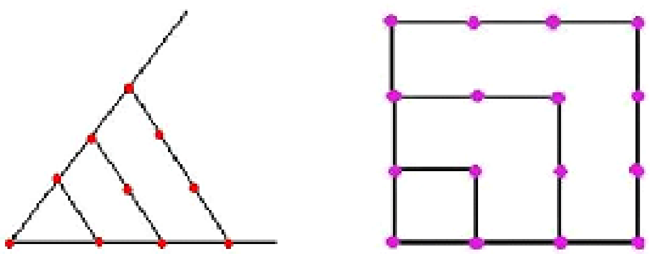

Fig. 1-2. Triangular and quadrate numbers

According to common definition, a number is called *K*-angular if it is one of the sums of the members of the arithmetical sequence with the first member equals 1 and the difference d = *K*-2 (for instance, [1]). All polygonal numbers can be put in the infinite table with the general formula of the *n*-th index of the series of *K*-angular numbers (Fig.3).

D	Polygon	Numbers						General member Sn
		S1	S2	S3	S4	S5	…	
1	Triangle	1	3	6	10	15	…	n(n+1)/2
2	Quadrate	1	4	9	16	25	…	n^2
3	Pentagon	1	5	12	22	35	…	n(3n-1)/2
4	Hexagon	1	6	15	28	45	…	n(2n-1)
…	…	…	…	…	…	…	…	…
D	…	1	2+d	3+3d	4+6d	5+10d	…	n(dn-(d-2))/2

Fig. 3. The table of polygonal numbers of different angularity

So, the general formula of *K*-angular number with index *n* is:

$$K_n = \frac{n\,[dn-(d-2)]}{2}$$

In order to achieve the objectives of our work we used the method of analysis of the polygonal numbers table (Fig. 3), known Diophantus's relation for triangular and quadrate numbers [6], methods of algebraic transformations, basics of the algebraic systems theory [5] etc. Apart from the theoretical methods we also applied instruments of MS Excel and GeoGebra software to build and investigate relevant computer models of polygonal numbers and their series. Detailed description of some tools will be given below at the specific points of the paper.

Main mathematical results of the paper make some relations among flat polygonal numbers obtained by the author basing on their properties investigation. In particular, on the base of analysis of *K*-angular numbers with the same index ***n*** the general formula of difference between *K*-angular and (*K-1*) - angular number with index ***n*** (n>0) was obtained:

$$R_n = n(n-1)/2 \qquad (1)$$

Based on this formula we formulated general formula of calculation of any K-angular number with index ***n*** via n-th triangular number and definite quantity (settled amount) of differences R_n. We noticed that the quantity of adding differences R_n depends upon the angularity *K* and makes *K-3*. So, required general formula of calculation of any ***n***-th K-angular number is:

$$K_n = 3_n + (K-3)R_n = 3_n + (K-3)n(n-1)/2 \qquad (2)$$

The relation between ***n***-th (n>2) pentagon number and quadrate number with index ***n–1*** was obtained on the base of analysis of series of pentagon and quadrate numbers. We have established such regularity. In order to obtain ***n***-th (n>2) pentagonal number to quadrate number indexed ***(n–1)*** it must be added natural number 4 and the sum ***S*** of an arithmetical progression with the first member equals 4 and difference d=1 which is calculated under the formula:

S = (n–2)(5+n)/2, where ***n*** is pentagonal number's index.

So, relation between ***n***-th (n>2) pentagon number and quadrate number indexed ***(n–1)*** is:

$$5_n = 4_{n-1} + 4 + (n-2)(5+n)/2, \qquad (3)$$

where 5_n is a pentagonal number indexed ***n***, and 4_{n-1} is a quadrate number indexed ***(n–1)***.

Some relations were obtained on the base of the known Diophantus's relation for triangular and quadrate numbers. In particular, the connection between indexes ***n*** and ***m*** of triangular 3_n and quadrate 4_m numbers accordingly was established for Diophantus's relation 8T+1=К (T – triangular number, К- quadrate number) (Little, 1910). So, Diophantus's formula is generalized now in such a relation:

$$8 \cdot 3_n + 1 = 4_m, \text{ where } m = 2n+1 \quad (4)$$

Geometrically it can be represented this way: the dots matching (in their amount) a quadrate number indexed m=2n+1 form quadrate. One point is in the center of the quadrate, and the rest of dots are grouped together in the form of 8 rectangular triangles with broken hypotenuses. Each triangle contains the amount of dots matching a triangular number with index ***n*** (Fig.4).

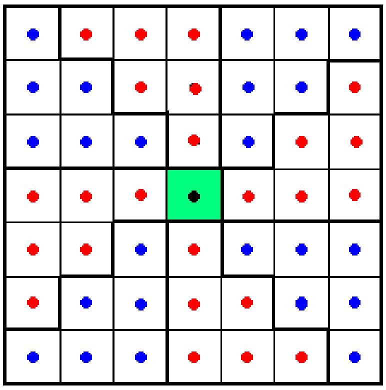

Fig. 4. Geometrical representation of relation (4) for n=3, m=7

Furthermore, intuitively from geometrical ideas it is clear that if we take some quantity of equal quadrates, then we can make another quadrate of them, i.e. we can obtain another quadrate number. So, the task is to define this quantity of the quadrates which make this quadrate number. Using algebraic transformations we found this quantity and as a result generalized the formula (4) to the following relation:

$$4_p(8 \cdot 3_n + 1) = 4_{p(2n+1)} \quad (5)$$

In geometrical sense it means that if we take a quadrate number indexed ***m=2n+1*** (see formula (4)) in amount matching quadrate number 4_p, we will receive a quadrate number indexed ***p(2n+1)***.

Based on the relation (4), analysis of the table of triangular and pentagon numbers, and from geometric reasons the relation between triangular and pentagon numbers was obtained. Really, intuitively it is clear that from a quadrate and four triangles we can build a pentagon. We checked this fact algebraically for triangular and pentagon numbers with different indexes and discovered the following regularity:

$$8 \cdot 3_n + 1 + 4 \cdot 3_n = 5_{(2n+1)} + n \quad (6)$$

From here the relation between triangular and pentagon numbers appears:

$$12 \cdot 3_n + 1 - n = 5_{(2n+1)} \quad (7)$$

Geometrically it means that from 12 triangles, each of which contains the amount of dots matching ***n***-th triangular number, and one more dot we can build a pentagon. It will contain the amount of dots matching a pentagon number indexed ***2n+1***. However, ***n*** dots will be extra, i.e. they will not be included into construction of the pentagon. We suggest a reader independent trying.

All relations (1-7) were proved with algebraic methods.

Determination of binary algebraic operations on the sets of polygonal numbers of various angularities. The following results were obtained on the base of the algebraic systems theory and the analysis of the polygonal numbers series.

We considered polygonal numbers of the angularity К≥3 as infinite sets and based on the definition of binary operation on a set. As a result, we obtained the following: at addition of two arbitrary *K*-angular numbers $К_n$ и $К_m$ in order to receive the third number of the same angularity it is necessary to the result of the traditional addition to add the product (*K*-2)mn, where K-2 = d what is difference of arithmetic progression. In order to prove this we used common formula of *K*-angular number with indexed **n** and **m** respectively.

So, on the sets of polygonal numbers of angularity K (К≥3) the following addition operation $\oplus$ is defined:

$$К_n \oplus К_m = К_n + К_m + (K-2)mn = К_{m+n},$$

where n, m$\in$N – indexes of *K*-angular numbers in a corresponding set.

The result of the obtained addition operation is the number of the same angularity with an index equal the sum of items' indexes. Thus, the binary operation of addition is invariant in relation to the angularity of numbers.

Proceeding from the definition of binary operation on a set, using also common formula of *K*-angular number and algebraic transformations we initially defined corresponding operations of multiplication on each set of polygonal numbers of angularity *K*=3,4,5,6,7,8. Then we found out regularity in the operations and obtained generalized formula which defines operation of multiplication for numbers of arbitrary angularity *K* with indexes n, m$\in$N:

$$K_n \otimes K_m = 2K_n \cdot K_m \frac{dnm-(d-2)}{(dm-(d-2))(dn-(d-2))} = K_{nm}$$

The result of the defined multiplication operation is the number of the same angularity with an index equal the product of factors' indexes. Thus, the binary operation of multiplication is invariant in relation to the angularity of numbers.

As on the sets of polygonal numbers of angularity *K* (*K*≥3) binary operations of addition and multiplication are defined these sets can be considered algebraic systems.

We also proved that the defined operations of addition and multiplication submit to commutative, associative and distributive laws.

The attempt of obtaining of the polygonal number with opposite number was taken. It was established that a polygonal number with an index ***-n*** equals a polygonal number indexed ***n+1***.

Investigation of algebraic properties of the polygonal numbers sets as algebraic systems. It was also considered the issue as for if the sets of polygonal numbers are groups, rings, and fields concerning the defined by us operations of addition and multiplication.

As a result, it was investigated and established that the set of polygonal numbers of angularity *K* (К≥3) is not a group in relation to the binary operation of addition as there is not a neutral element 0 in it.

It was established that the sets of polygonal numbers of angularity *K* (К≥3) are not groups in relation to the binary operation of multiplication as there is no a symmetrical element to elements of these sets.

It was proved also that the set of polygonal numbers of angularity K (К≥3) is not a ring in relation to the binary operations of addition and multiplication as there is no a reversibility of addition on the sets.

Computer modeling of flat polygonal numbers and educational prospects. Obtained results and algebraic properties of flat polygonal numbers pushed us to use them as fruitful material for involving students into research and constructive process of mathematics learning.

First of all we constructed author's dynamic models of polygonal numbers in GeoGebra environment and developed special didactic support in order to encourage students to make independent conclusions as for these numbers formation and composition of their series.

For example, we can suggest students answering such a chain of questions based on the work with the models (Fig.5):

1) What happens with amount of dots on each side of triangle? – Using the slider students have to conclude that each side on each step is divided by one more dot.

2) How many dots on each step are added in total? Why? – It is worthwhile to push students to realize that each next polygonal number contains all points obtained on previous step (Fig.5).

3) Using the model write down 5 sequent triangular (quadrate, pentagon,…) number.

4) Can you check your hypothesis as for division of each side of polygon for quadrate numbers, for pentagons numbers and so on?

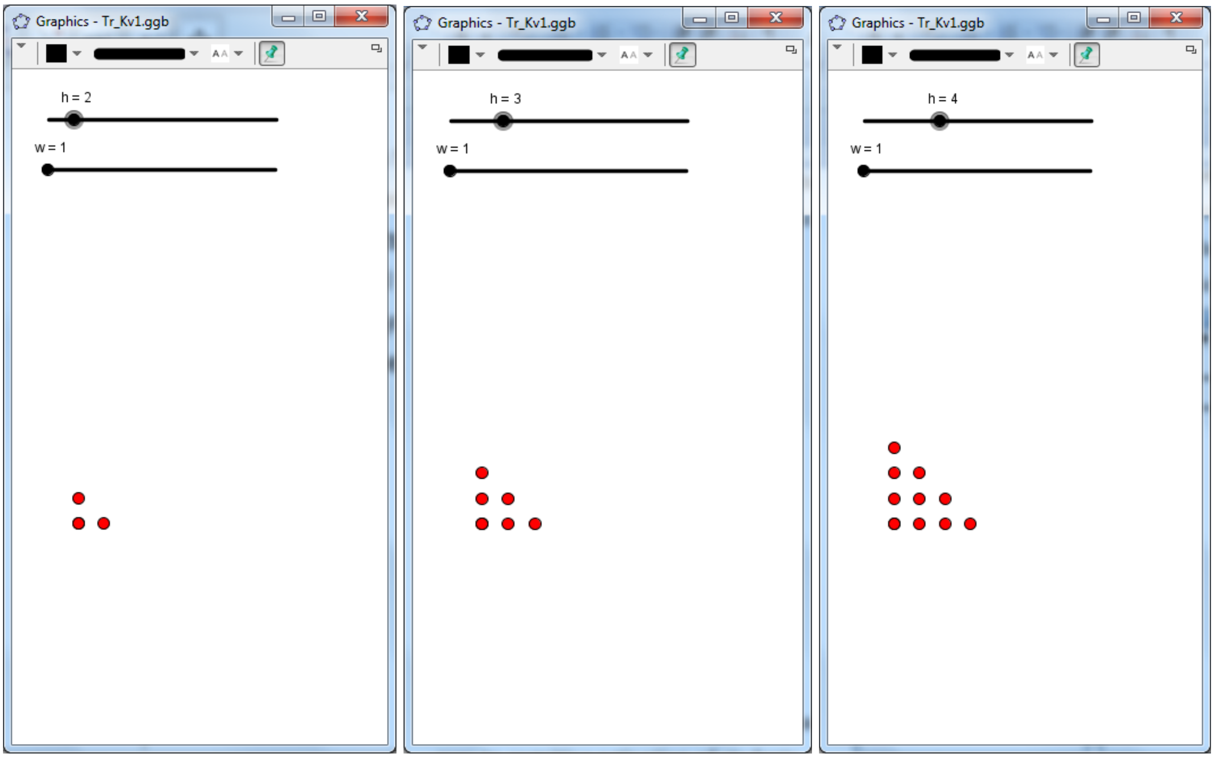

Fig. 5. Step-by-step work with a dynamic model

So, we can allow students to combine in one model numbers of different angularities and "play" with them to check obtained regularities and find out their own ones, and then – to prove them algebraically (Fig. 6).

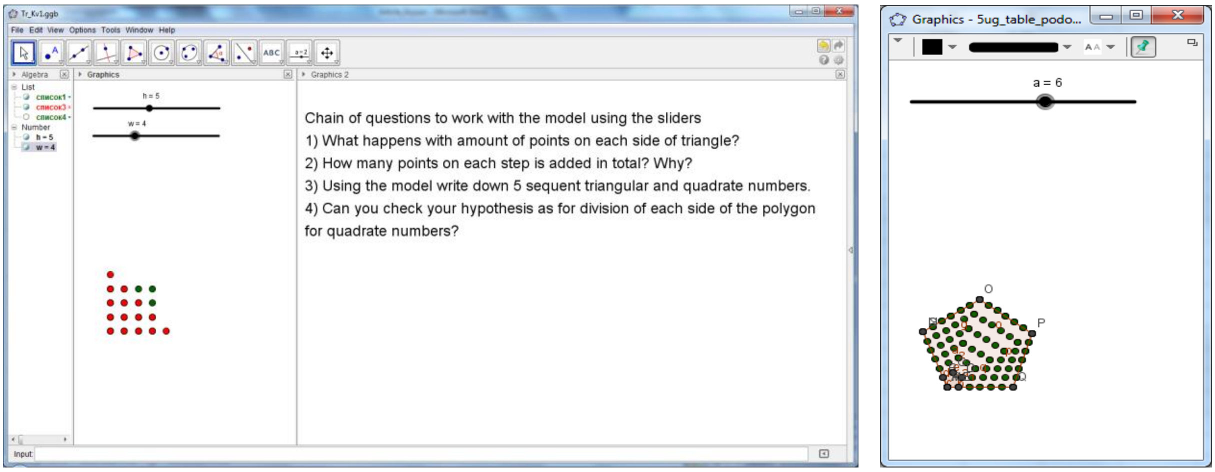

Fig. 6. Episodes of work with dynamic models of various polygonal numbers in GeoGebra environment

Experimental work with ready-made models of polygonal numbers is only one way to cognitive activation of students. We would recommend also another one, so called constructive way, to use polygonal numbers phenomena in order to progress students' motivation. It is based on the idea of encouraging students to build similar computer models themselves.

The matter is that it is easy to obtain the total amount of points algebraically. However, it might be a challenge for students to locate proper amount of dots in the proper shape of a regular polygon and to automatize this location using GeoGebra tools or to develop a computer program to obtain proper polygons. More over, polygons must be geometrically similar and they must be overlaid in order to contain total amount of dots. Here a student bumps into necessity to find out regularity again and to find out the way how to visualize this regularity using computer.

We would suggest a sort of pre-teaching in order to push students to their "discovery". So, we again offer the chain of questions:

1) If we have a point coordinated (***m,n***), how should we change these coordinates in order to make them form on each step new quadrate (triangular) number?

2) What will happen if we fix ***m*** and change ***n*** with some step?

3) What are limits in which we have to change ***m*** and ***n***?

As a result, students come to the conclusion that for a quadrate number there is such a regularity: if we fix ***n***, ***m*** must change its value from 0 to P-1, where P is the index of a quadrate number, and this must be repeated for each ***n*** which itself changes again from 0 to P-1.

In the process of creating the triangular numbers model it is important to ask students as for the connection of the coordinate ***m*** and final value of ***n***.

Here we can also relevantly introduce Sequence – function as an important built-in GeoGebra tool, its nested variant, and demonstrate its application. As well we can provide knowledge integration with algorithms and programming (repetition structure, loop operator): both elicit this analogy from students and then stress it.

Finally, it is also really beneficial to encourage students to build universal model for polygonal numbers of any angularity as it is connected with some practical geometrical problems of points' location on the plane. Here we would offer students the following chain of subtasks.

1) The task of placing ***K*** dots around the circle, where **K** is the angularity of a polygonal number (to elicit from students the proper angle of the rotation and the list of obtained rotations).

2) The task of repetition of the list of the obtained rotations according to the index of a polygonal number of proper angularity.

3) The task of division of each segment between two dots on proper amount of points according to the index of a polygonal number.

As a result, students come to the nested construction of sequences and elements of lists, realize it using built-in Sequence-function and List-function of GeoGebra and can manipulate it to verify some theoretical regularities among polygonal numbers (Fig.7).

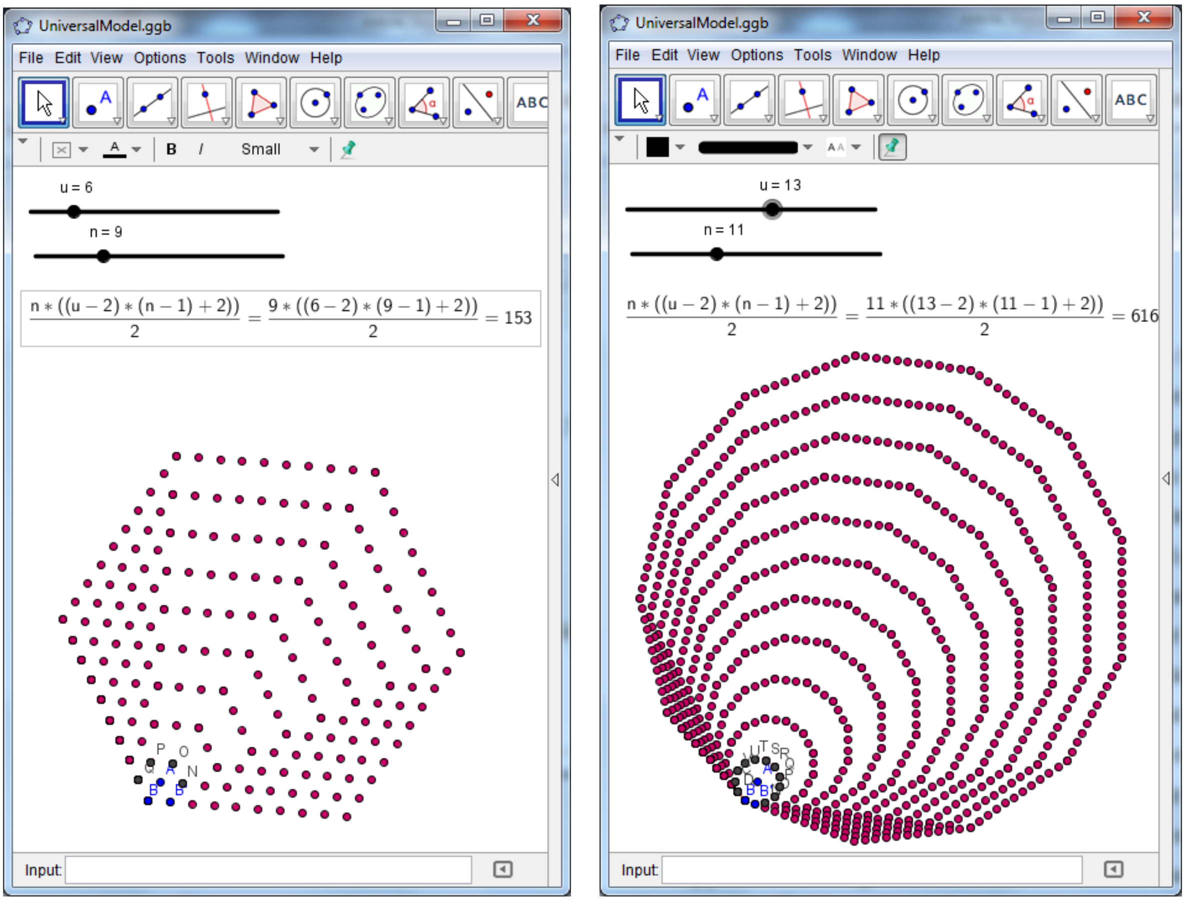

Fig. 7. Episodes of students' work in GeoGebra with built universal dynamic model for polygonal numbers of any angularity

In addition, we can encourage students to build two universal tables (using spreadsheets, for example MS Excel) in order to verify and demonstrate the action of binary algebraic operations of addition and multiplication on the sets of polygonal numbers of various angularities determined above. As next step of work it might be students' investigation of geometrical interpretation of the algebraic operations.

After manipulations with the universal model and tables of the binary operations built by students we can emphasize nowadays applications of polygonal numbers, and get students familiar with those practically urgent tasks where polygonal numbers themselves and their constructed dynamical models might be used. Among these applications and tasks it is relevant to mention at least the following ones: (1) the problem of determination of maximum amount of equal figures located inside of specific shape and its derivation – the problem of dense package of balls inside the shape [7]; (2) method of time series smoothing based on polygonal numbers usage and applied to investigation of indicators of funds markets [2]; (3) genetic algorithms modified basing on polygonal numbers [4]; (4) location of cell connection towers and others.

Conclusions. According to the aims of the work, there were obtained and represented some relations among flat polygonal numbers based on their properties investigation; binary algebraic operations on the sets of polygonal numbers of various angularity were determined and algebraic properties of the polygonal numbers sets as algebraic systems were investigated; author's computer models of polygonal numbers were built in GeoGebra environment and the ways in which these and other results might be used in mathematical education were determined. In a whole, obtained results may be considered as didactic support of Mathematics and Computer Science learning.

Using this didactic support we will be able to provide some benefits for students.

The most important of them seem to be bringing up students' holistic view at our world, integrative understanding of algebra and geometry as well as realizing integrative role of Mathematics as a universal language for reality description. Another educational benefit is forming students' philosophical and historical outlook as polygonal numbers are presented to students as a really bright example of historical continuity. They can trace historical roots of the concept itself and transformations of its practical applications.

Apart from integrative and philosophical outlook, with the help of our didactic tools we will be able to bring up and develop students' skills of recurrent relations obtaining, necessary for their algorithmic thinking forming. We can also develop students' modeling skills as we encourage them to build dynamic computer models of mathematical objects and to manipulate them. As a result, we can expect raising of students' motivation and interest to science learning and exploration.

The didactic tools presented above might be used both in curriculum and extra-curriculum studying of Mathematics and Computer Science in senior grades of schools. They also can be interesting for teachers including pre-service ones.

The prospects of our work are empirical proving of our theoretical predictions as for the influence of our didactic tools on the students' skills and characteristics.

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Анотація. Гризун Л.Е. Комп'ютерні моделі для дослідження багатокутних чисел та їх застосування у математичній освітній практиці.

Сучасні потреби суспільства у зростанні рівня математичної освіти, недостатнє застосування когнітивного потенціалу математики як науки і одночасно падіння мотивації школярів та студентів до її вивчення актуалізують пошук адекватних дидактичних механізмів, залучення інформаційних технологій, а також необхідність підвищення педагогічного потенціалу фундаментальних математичних понять. У цьому контексті феномен багатокутних чисел являє яскравий приклад когнітивного математичного і одночасно дидактичного ресурсу, спонукає вивчення їх властивостей. Відповідно до цілей роботи, було одержано деякі залежності між пласких багатокутних чисел на основі дослідження їх властивостей; визначено бінарні алгебраїчні операції на множинах багатокутних чисел різної кутності та досліджено алгебраїчні властивості означених чисел як алгебраїчних систем; побудовано авторські комп'ютерні моделі багатокутних чисел у середовищі GeoGebra; визначено шляхи застосування цих моделей в освітній практиці.

Ключові слова: *плаські багатокутні числа та їх властивості; комп'ютерні динамічні моделі; математична освіта.*

Аннотация. Гризун Л.Э. Компьютерные модели для исследования многоугольных чисел и их применение в математической образовательной практике.

Современные потребности общества в подъеме уровня математического образования, недостаточное использование когнитивного потенциала математики как науки и одновременное снижение мотивации школьников и студентов к ее изучению актуализируют поиск адекватных дидактических механизмов, применение информационных технологий, а также необходимость повышения педагогического потенциала фундаментальных математических понятий. В этом контексте феномен многоугольных чисел представляет собой яркий пример когнитивного математического и одновременно дидактического ресурса, актуализирует исследование их свойств. В работе получены некоторые зависимости между плоскими многоугольными числами на основе исследования их свойств; определены бинарные операции на множествах многоугольных чисел различной угольности и исследованы алгебраические свойства данных чисел как алгебраических систем; построены авторские компьютерные модели многоугольных чисел в среде GeoGebra; предложены пути применения данных моделей в образовательной практике.

Ключевые слова: *плоские многоугольные числа и их свойства; компьютерные динамические модели; математическое образование.*

Abstract. Gryzun L.E. Computer models for polygonal numbers investigation and their use in mathematical education.

Urgent needs of society to raise level of mathematical education and falling students' motivation to learn mathematics as well as insufficient using of its cognitive power in the *educational process actualizes necessity to propose relevant didactic mechanisms, to apply computer technologies and make mathematical concepts serve educational purposes. In this context the phenomenon of polygonal numbers represents really bright example of both cognitive resource of math and its application to didactics and real life. According to the aims of the work, there were obtained and represented some relations among flat polygonal numbers based on their properties investigation; binary algebraic operations on the sets of polygonal numbers of various angularity were determined and algebraic properties of the polygonal numbers sets as algebraic systems were investigated; author's computer models of polygonal numbers were built in GeoGebra environment and the ways in which these and other results might be used in mathematical education were determined.*

Kew words: *flat polygonal numbers and their properties; computer dynamic models; mathematical education.*